# Assignment 9

Hand in no. 4, 5, 7 and 8 November 21.

1. Consider the problem of minimizing $f(x,y,z) = (x+1)^2 + y^2 + z^2$ subjecting to the constraint $g(x,y,z) = z^2 - x^2 - y^2 - 1,\ z > 0$. First solve it by eliminating $z$ and then by Lagrange multipliers.

2. Let $f, g_1, \cdots, g_m$ be $C^1$-functions defined in some open $U$ in $\mathbb{R}^{n+m}$. Suppose $(x_0, y_0)$ is a local extremum of $f$ in $\{(x,y) \in U : \ g_1(x,y) = \cdots = g_m(x,y) = 0\ \}$. Assuming that $D_y G(x_0, y_0)$ is invertible where $G = (g_1, \cdots, g_m)$, show that there are $\lambda_1, \cdots, \lambda_m$ such that
$$\nabla f + \lambda_1 \nabla g + \cdots + \lambda_m \nabla g_m = 0\ ,$$
at $(x_0, y_0)$.

3. Solve the IVP for $f(t,x) = \alpha t(1+x^2), \alpha > 0,\ t_0 = 0$, and discuss how the (largest) interval of existence changes as $\alpha$ and $x_0$ vary.

4. Let $f \in C(R)$ where $R$ is a closed rectangle. Suppose $x$ solves $x' = f(t,x)$ for $t \in (a,b)$ with $(t, x(t)) \in R$. Show that $x$ can be extended to be a solution in $[a,b]$.

5. Let $f \in C(R)$ where $R$ is a closed rectangle satisfy a Lipschitz condition in $R$. Suppose that $x$ solves $x' = f(t,x)$ for $t \in [a,b]$ where $(b, x(b))$ lies in the interior of $R$. Show that there is some $\delta > 0$ such that $x$ can be extended as a solution in $[a, b+\delta]$.

6. Provide a proof to Theorem 3.15 (Picard-Lindelof theorem for systems).

7. Let $\mathbf{e}_n = (0, \cdots, 0, 1, 0, \cdots, )$ by the sequence with 1 at the $n$-th place and equal to 0. Consider the sequence formed by these $\mathbf{e}_j$'s. Show that it has no convergent subsequences in the space $l^p,\ 1 \le p \le \infty$ Recall that $l^p$ is the space consisting of all sequences $\mathbf{a} = \{a_n\}$ satisfying $\|\mathbf{a}\|_p = (\sum_n |a_n|^p)^{1/p} < \infty$ and $\|\mathbf{a}\|_\infty = \sup_n |a_n|$.

8. Consider $\{f_n\}, f_n(x) = x^{1/n}$, as a subset in $C[0,1]$. Show that it is a closed, bounded, but has no convergent subsequence in $C[0,1]$.

9. Prove that $\{\cos nx\}_{n=1}^{\infty}$ does not have any convergent subsequence in $C[0,1]$.

10. Show that any finite set in $C(\overline{G})$ is bounded and equicontinuous.